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1990 J. Phys. A: Math. Gen. 23 5241

(http://iopscience.iop.org/0305-4470/23/22/015)

J. Phys. A: Math. Gen. **23** (1990) 5241–5256. Printed in the UK

Chern–Simons quantum mechanics

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Received 13 February 1990

Abstract. We consider the quantum mechanics coupled to the gauge field which has the Chern–Simons term as the action. The generalized Hamiltonian formalism of the system is constructed and canonical and path-integral quantizations are performed. In solving the Gauss law constraint, the gauge field is represented by a multivalued function and the charge density operator of the particle. The gauge potential is singular and also induces a singular magnetic field though the magnetic flux is finite. By a phase transformation, the singular gauge field is eliminated. Then the transformed wavefunction absorbing the gauge field induces a novel phase factor under 2π-rotation in space. Thus we show that the transformed wavefunction describes exotic spin state. The extension of these analyses to the relativistic case is also presented.

1. Introduction

The Chern–Simons term is making a strong impact on physics [1]. Several types of quantum field theories with the Chern–Simons term in action [2] have been investigated up to now. Recently, Dzyaloshinski *et al* and Polyakov *et al* [3,4] analysed the CP^1 model with the Chern–Simons term as the action of the hidden U(1) gauge field, which might be an effective theory explaining high-T_c superconductivity phenomena [5]. On the other hand, Witten [6] has investigated the quantum field theory with the pure Chern–Simons term, which gives a field-theoretical framework to understand knot theory. Stimulated by these works, many investigations of quantum field theories with the Chern–Simons term have appeared.

The spin and statistics in (2+1) dimensions have a specific feature. We cannot completely determine them from the algebra of the angular momentum operator. There remains an ambiguity in the representation of the operator algebra because the algebra has the nature of the Abelian group. It may allow us to expect the appearance of unusual spin and statistics in (2+1)-dimensional quantum theories. Some years ago, Wilczek specified that such exotic spin and statistics appeared in quantum mechanics under suitable background fields [7]. Such an exotic state is called an *anyon*. After that, some analyses appeared [8] and some interesting physics has been pointed out [9].

In the paper by Polyakov [4], the phenomenon called Bose–Fermi transmutation is presented in the elegant path integral formalism, where the CP^1 model with the

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Chern–Simons term as the action of the gauge field is treated. Before that, Wilczek and Zee [10] considered the CP^1 model with the Hopf term and showed that the soliton discovered by Belavin and Polyakov [11] has exotic statistics. The model of Polyakov is the low-energy effective theory of the model of Wilczek and Zee.

Before and since then, there have been some interesting works related to the above. In the procedure of canonical quantization, the O(3) nonlinear σ model with the Hopf term, the massive scalar theory with the Chern–Simons term and the CP^1 model with the Chern–Simons term have been discussed [12]. These topological terms are considered as the Wess–Zumino term in (2+1) dimensions [13]. Relations between Polyakov's regularization [4], the anyon and the braid group have been discussed [14]. In the connection with the statistical model, it has been investigated as to how the topological term can be induced from the antiferromagnetic Heisenberg model or the Hubbard model in the long-wavelength limit [15]. The importance of the role of the P- and T-violating term has been specified [16]. The analyses of several types of quantum field theories with the Hopf term or the Chern–Simons term are now in progress [17].

We have investigated the fermion-coupled CP^1 model with the Chern–Simons term [18]. It has been shown that the statistics of each matter field has been transmuted to an exotic one by investigating the canonical commutator algebra. The same result has also been obtained in the generic U(1) gauge field theories which have the Chern–Simons term as the action of the gauge field, without specifying the detail of the matter fields [19]. A similar analysis has also been carried out in the context of motivating the bosonization in (2+1) dimensions [20].

In this paper, we investigate quantum mechanics in which the gauge field has the Chern–Simons term as the action. We construct the generalized Hamiltonian formalism of the theory and quantize it by using the canonical and path-integral methods. The Gauss law constraint is solved and the gauge field is represented by using a multivalued function and the charge density operator of the particle. The gauge field is singular and also gives a singular magnetic field, although the magnetic flux is finite. We eliminate the singular gauge field by phase transformation of the wavefunction. We then show that the transformed wavefunction induces a novel phase factor under the rotation in two-dimensional space. Thus the state described by the transformed function has the exotic spin. Further we extend these analyses to the case of the relativistic particle and show that the exotic spin state appears in this case.

One of our aims is to make clear the essential feature of the appearance of the exotic spin state and the Bose–Fermi transmutations in the rather simplified model. How is it derived? We also hope that the model will be of use in a realistic system. The model can be used to describe a system where field-theoretical effects such as vacuum polarization, are not important. The model also gives an example of the system including anyons [7]. Part of our results has already been reported [21].

This paper is organized as follows. In section 2, we formulate the generalized Hamiltonian formalism and the canonical and path-integral quantizations are presented. In section 3, we show that the exotic spin state appears in the system. The extension to the relativistic case is given in section 4. Section 5 is devoted to discussion and conclusions.

2. Generalized Hamiltonian formalism and quantizations

In this section, we set up the consistent quantum mechanics coupled to the U(1) gauge field, which has the Chern–Simons term as the action. Following the Dirac algorithm [22], the generalized Hamiltonian system is constructed and then we quantize it canonically. The path-integral expression is also given following the Faddeev–Senjanovic formula [23].

2.1. Dirac Formalism

The Lagrangian of the non-relativistic particle coupled to the U(1) gauge field with the Chern–Simons term as the action is given by

$$L = -\frac{m}{2}\dot{q}_i(t)\dot{q}^{\,i}(t) + \int \mathrm{d}\boldsymbol{x}\,\{eA_0(t,\boldsymbol{x})\delta(\boldsymbol{x}-\boldsymbol{q}) + eA_i(t,\boldsymbol{x})\delta(\boldsymbol{x}-\boldsymbol{q})\dot{q}^i + \theta\varepsilon^{\mu\nu\rho}A_\mu(t,\boldsymbol{x})\partial_\nu A_\rho(t,\boldsymbol{x}) \tag{2.1}$$

where m is the mass of the particle and e is the gauge coupling constant.

The last term in (2.1) is the so-called Chern–Simons term. θ is a parameter which plays an important role in subsection 3.2. The dot denotes the time derivative. From the Lagrangian (2.1), we can derive the Euler–Lagrange equations,

$$m\ddot{q}_i - eF_{0i} - eF_{ji}\dot{q}^j = 0 \tag{2.2a}$$

$$2\theta\varepsilon^{\alpha\beta\mu}\partial_\alpha A_\beta + eg^{0\mu}\delta(\boldsymbol{x}-\boldsymbol{q}) + eg^{i\mu}\delta(\boldsymbol{x}-\boldsymbol{q})\dot{q}_i = 0 \tag{2.2b}$$

where $F_{0i} = \mathrm{d}A_i/\mathrm{d}t - \partial A^0/\partial q^i$ and $F_{ji} = \partial A_i/\partial q^j - \partial A_j/\partial q^i$.

We consider q^i $(i = 1,2)$ and $A^\mu(t,\boldsymbol{x})$ $(\mu = 0,1,2)$ as the dynamical variables. The short-hand notation $A^\mu(t,\boldsymbol{q})$ may be used. We then recognize it as $A^\mu(t,\boldsymbol{q}) = \int \mathrm{d}\boldsymbol{x}\, A^\mu(t,\boldsymbol{x})\delta(\boldsymbol{x}-\boldsymbol{q})$. Further, we omit the time variable t from the argument where there can be no confusion.

The canonical momenta for them are

$$p^i \equiv \frac{\partial L}{\partial \dot{q}_i} = -m\dot{q}^i + eA^i(\boldsymbol{q}) \tag{2.3a}$$

$$\Pi^\mu \equiv \frac{\partial L}{\partial \dot{A}_\mu} = \theta\varepsilon^{\alpha 0\mu}A_\alpha . \tag{2.3b}$$

Equation (2.3a) gives the velocity by the momentum as

$$\dot{q}^i = -\frac{1}{m}\left(p^i - eA^i(\boldsymbol{q})\right). \tag{2.4}$$

Equation (2.3b) shows that there is the primary constraint

$$\phi^\mu \equiv \Pi^\mu - \theta\varepsilon^{\alpha 0\mu}A_\alpha \approx 0 \tag{2.5}$$

where the weak equality is recognized as usual [22]. The Poisson bracket for the dynamical variables are defined as

$$\{A_\mu(\boldsymbol{x}), \Pi_\nu(\boldsymbol{y})\} = g_{\mu\nu}\delta(\boldsymbol{x}-\boldsymbol{y}) \tag{2.6a}$$

$$\{q^i, p^j\} = g^{ij} . \tag{2.6b}$$

By the Legendre transformation, the canonical Hamiltonian is obtained formally as

$$\begin{aligned} H_{\mathrm{C}} &= \dot{q}_i p^i + \int \mathrm{d}\boldsymbol{y}\, \dot{A}_\mu \Pi^\mu - L \\ &= -\frac{1}{2m}(p^i - eA^i(\boldsymbol{q}))(p_i - eA_i(\boldsymbol{q})) - eA_0(\boldsymbol{q}) - \int \mathrm{d}\boldsymbol{y}\, A_0 2\theta\varepsilon^{0ij}\partial_i A_j . \end{aligned} \tag{2.7}$$

Now, following Dirac's prescription, we construct the consistent Hamiltonian formalism. At first we prepare the primary Hamiltonian,

$$H_{\mathrm{P}} = H_{\mathrm{C}} + \int \mathrm{d}\boldsymbol{y} \;\; u_\rho \phi^\rho \tag{2.8}$$

which includes the primary constraint (2.5) with the Lagrange multiplier u_ρ. We require the consistency condition that a constraint is not time-evolved by the primary Hamiltonian H_{P}. This condition might (i) be satisfied, or (ii) determine a part of the multipliers, or (iii) induce a new constraint. In case (iii), we require the consistency for the new constraint again and continue this algorithm until all constraints reduce to case (i) or (ii). We then can find the consistent Hamiltonian system which gives the evolution of the dynamical variables in the true phase space.

The consistency condition for ϕ^μ is

$$\dot{\phi}^\mu = \{\phi^\mu, H_{\mathrm{P}}\} \approx 0$$

which gives

$$\begin{aligned} -g_i^\mu \frac{e}{m}(p^i - eA^i(\boldsymbol{q}))\delta(\boldsymbol{x}-\boldsymbol{q}) + g_0^\mu e\delta(\boldsymbol{x}-\boldsymbol{q}) + g_0^\mu 2\theta\varepsilon^{0ij}\partial_i A_j(\boldsymbol{x}) \\ - g_j^\mu 2\theta\varepsilon^{0ij}\partial_i A_0(\boldsymbol{x}) - 2\theta\varepsilon^{0\mu\rho}u_\rho \approx 0 \end{aligned}$$

where the Poisson brackets (2.6) are used. When $\mu = 0$, we have the secondary constraint

$$\phi^0_{(2)} \equiv 2\theta\varepsilon^{0ij}\partial_i A_j(\boldsymbol{x}) + e\delta(\boldsymbol{x}-\boldsymbol{q}) \approx 0 \tag{2.9}$$

which is the Gauss law constraint and plays a important role in subsection 3.1. When $\mu = k\,(=1,2)$, the condition is

$$-\frac{e}{m}(p^k - eA^k(\boldsymbol{q}))\delta(\boldsymbol{x}-\boldsymbol{q}) - 2\theta\varepsilon^{0ik}\partial_i A_0(\boldsymbol{x}) - 2\theta\varepsilon^{0kj}u_j \approx 0 \tag{2.10}$$

which determines u_j.

Next we consider the consistency condition for $\phi^0_{(2)}$

$$\dot{\phi}^0_{(2)} = \{\phi^0_{(2)}, H_{\mathrm{P}}\} \approx 0.$$

From this, we obtain

$$-\frac{e}{m}(p^i - eA^i(\boldsymbol{q}))\frac{\partial}{\partial q^i}\delta(\boldsymbol{x}-\boldsymbol{q}) + 2\theta\varepsilon^{0ij}\partial_i u_j \approx 0$$

Table 1. The Poisson brackets among the constraints.

	ϕ^0	ϕ^1	ϕ^2	$\phi^0_{(2)}$
ϕ^0	0	0	0	0
ϕ^1	0	0	$-2\theta\delta(\boldsymbol{x}-\boldsymbol{y})$	$2\theta\partial^y_2\delta(\boldsymbol{x}-\boldsymbol{y})$
ϕ^2	0	$2\theta\delta(\boldsymbol{x}-\boldsymbol{y})$	0	$-2\theta\partial^y_1\delta(\boldsymbol{x}-\boldsymbol{y})$
$\phi^0_{(2)}$	0	$-2\theta\partial^x_2\delta(\boldsymbol{x}-\boldsymbol{y})$	$2\theta\partial^x_1\delta(\boldsymbol{x}-\boldsymbol{y})$	0

which is satisfied if (2.10) is used. Therefore $\phi^0_{(2)}$ is consistent.

Now we obtain the consistent set of the constraints (2.5) and (2.9). These constraints are classified into first- and second-class constraints. First-class constraints are those where the Poisson brackets with all other constraints vanish. Second-class constraints are all others. The Poisson brackets between each constraint are given by table 1.

From table 1, we find that ϕ^0 is the first-class constraint and all others are second-class ones. The number of second-class constraints is three, i.e. an odd number, so that all the second-class constraints are not independent. We modify the Gauss law constraint taking the linear combination among the second-class constraints as

$$G = \phi^0_{(2)} - \int d\boldsymbol{w}_1\, d\boldsymbol{w}_2 \{\phi^0_{(2)}, \phi^s(\boldsymbol{w}_1)\}(C^{-1})_{st}(\boldsymbol{w}_1,\boldsymbol{w}_2)\phi^t(\boldsymbol{w}_2) \tag{2.11}$$

where $C_{st}(\boldsymbol{w}_1,\boldsymbol{w}_2) = \{\phi_s(\boldsymbol{w}_1), \phi_t(\boldsymbol{w}_2)\} = -2\theta\varepsilon_{0st}\delta(\boldsymbol{w}_1-\boldsymbol{w}_2)$, so that

$$(C^{-1})^{st} = \frac{1}{2\theta}\varepsilon^{0st}\delta(\boldsymbol{w}_1-\boldsymbol{w}_2). \tag{2.12}$$

G becomes the first-class constraint and the remaining second class constraints are ϕ^1 and ϕ^2.

At this stage, we construct the Dirac brackets which are defined as

$$\{A(\boldsymbol{x}), B(\boldsymbol{y})\}_{\mathrm{D}} \equiv \{A(\boldsymbol{x}), B(\boldsymbol{y})\} - \int \mathrm{d}\boldsymbol{w}_1 \mathrm{d}\boldsymbol{w}_2\, \{A(\boldsymbol{x}), \phi^s(\boldsymbol{w}_1)\}(C^{-1})_{st}(\boldsymbol{w}_1,\boldsymbol{w}_2) \times \{\phi^t(\boldsymbol{w}_2), B(\boldsymbol{y})\}$$

for any variables $A(\boldsymbol{x})$ and $B(\boldsymbol{y})$. Using the Dirac brackets, all the second-class constraints become the strong equations. After some calculations, we obtain

$$\{A_\mu(\boldsymbol{x}), A_\nu(\boldsymbol{y})\}_{\mathrm{D}} = \frac{1}{2\theta}\varepsilon_{0\mu\nu}\delta(\boldsymbol{x}-\boldsymbol{y}) \tag{2.13a}$$

$$\{A_0(\boldsymbol{x}), \Pi_0(\boldsymbol{y})\}_{\mathrm{D}} = \delta(\boldsymbol{x}-\boldsymbol{y}) \tag{2.13b}$$

$$\{A_i(\boldsymbol{x}), \Pi_j(\boldsymbol{y})\}_{\mathrm{D}} = \frac{1}{2}g_{ij}\delta(\boldsymbol{x}-\boldsymbol{y}) \tag{2.13c}$$

$$\{\Pi_\mu(\boldsymbol{x}), \Pi_\nu(\boldsymbol{y})\}_{\mathrm{D}} = \frac{\theta}{2}\varepsilon_{0\mu\nu}\delta(\boldsymbol{x}-\boldsymbol{y}) \tag{2.13d}$$

$$\{q^i, p^j\}_{\mathrm{D}} = g^{ij}. \tag{2.14}$$

Now our system is described by the total Hamiltonian

$$
\begin{aligned}
H_{\mathrm{T}} &= H_{\mathrm{C}} + \int \mathrm{d}\boldsymbol{y}\,(u\phi^0 + vG) \\
&= -\frac{1}{2m}(p^i - eA^i(\boldsymbol{q}))(p_i - eA_i(\boldsymbol{q})) + \int \mathrm{d}\boldsymbol{y}\,(u\phi^0 + v\phi^0_{(2)})
\end{aligned}
\tag{2.15}
$$

where we have used the strong equations and redefined v as $A_0 + v \to v$. We have obtained the consistent generalized Hamiltonian formalism of the non-relativistic particle coupled to the U(1) gauge field which has the Chern–Simons term as the action.

2.2. Canonical quantization

We have obtained the generalized Hamiltonian formalism for our system. The Dirac brackets are given by (2.13) and (2.14). The total Hamiltonian is (2.15) with the first-class constraints ϕ^0 and $\phi^0_{(2)}$.

Let us quantize this system canonically. The Dirac brackets are replaced by the equal-time commutation relations.

Therefore we have

$$[A_i(\boldsymbol{x}), A_j(\boldsymbol{y})]_- = \frac{1}{2\theta}\varepsilon_{ij}\delta(\boldsymbol{x}-\boldsymbol{y}) \tag{2.16a}$$

$$[A_0(\boldsymbol{x}), \Pi_0(\boldsymbol{y})]_- = \delta(\boldsymbol{x}-\boldsymbol{y}) \tag{2.16b}$$

$$[A_i(\boldsymbol{x}), \Pi_j(\boldsymbol{y})]_- = \frac{1}{2}g_{ij}\delta(\boldsymbol{x}-\boldsymbol{y}) \tag{2.16c}$$

$$[\Pi_i(\boldsymbol{x}), \Pi_j(\boldsymbol{y})]_- = \frac{\theta}{2}\varepsilon_{ij}\delta(\boldsymbol{x}-\boldsymbol{y}) \tag{2.16c}$$

$$[q^i, p^j]_- = g^{ij}. \tag{2.17}$$

In the above system, two first-class constraints remain. In order to restrict the system on the true phase space, we have to impose a subsidiary condition on the system. The condition should be chosen as satisfying an admissible condition. Thus det $\{\phi^0, \phi^0_{(2)}, f_1, f_2\}_{\mathrm{D}} \neq 0$, where f_1 and f_2 are the subsidiary conditions. The bracket $\{\phi^0, \phi^0_{(2)}, f_1, f_2\}_{\mathrm{D}}$ means the matrix whose elements are the Dirac brackets among $\phi^0, \phi^0_{(2)}$ and f_1, f_2. The admissible condition guarantees that the subsidiary conditions are independent of the first-class constraints and give the restriction on the true phase space. As an example, we adopt

$$f_1 = \partial_i A^i \approx 0 \tag{2.18}$$

$$f_2 = 2\theta\partial_j\partial^i A_0 + \frac{e}{m}\varepsilon_{kj}\partial_x^k\delta(\boldsymbol{x}-\boldsymbol{q})(p^j - eA^j(\boldsymbol{q})) \approx 0. \tag{2.19}$$

f_1 is the Coulomb gauge fixing condition and f_2 is determined as f_1 is consistent with the Euler–Lagrange equation (2.2*b*). These conditions restrict the dynamical variables to the physical ones.

2.3. Path-integral quantization

Based on the generalized Hamiltonian formalism presented before, we can perform the path-integral quantization following the Faddeev–Senjanovic method [23].

The partition function is given by

$$Z = N \int \mathrm{d}q_k \, \mathrm{d}p_k \, \mathrm{d}A_\mu \, \mathrm{d}\Pi_\mu \, \delta(\phi^0)\delta(\phi^0_{(2)})\delta(f_1)\delta(f_2) \det\{\phi^0, \phi^0_{(2)}, f_1, f_2\}_{\mathrm{D}}$$
$$\times \, \delta(\phi^1)\delta(\phi^2)\det\{\phi^1, \phi^2\}_{\mathrm{P}}$$
$$\times \exp\left[\mathrm{i}\left(\int \mathrm{d}t \, \dot{q}_i p^i + \int \mathrm{d}^3x \, \dot{A}_\mu \Pi^\mu - \int \mathrm{d}t \, H_{\mathrm{C}}\right)\right] \tag{2.20}$$

where N is a normalization factor. We can evaluate $\{\phi^0, \phi^0_{(2)}, f_1, f_2\}_{\mathrm{D}}$ and the result is given in table 2. The determinant of the matrix is independent of the dynamical variables and is included in the normalization factor N. The matrix $\{\phi^1, \phi^2\}_{\mathrm{P}}$ can be found from table 1 omitting $\phi^{(0)}$ and $\phi^0_{(2)}$. It also does not depend on the dynamical variables so that it is included in the normalization factor N. On further performing a part of the functional integration, equation (2.20) reduces to the formula

$$Z = \int \mathrm{d}q_k \, \mathrm{d}p_k \, \mathrm{d}A_i \, \delta\big(2\theta\varepsilon^{ij}\partial_i A_j(\boldsymbol{x}) + e\delta(\boldsymbol{x} - \boldsymbol{q})\big)\delta\big(\partial_i A^i(\boldsymbol{x})\big)$$
$$\times \exp\left\{\mathrm{i}\left[\int \mathrm{d}t\left(\dot{q}_i p^i + \frac{1}{2m}\big(p_i - eA_i(\boldsymbol{q})\big)\big(p^i - eA^i(\boldsymbol{q})\big)\right)\right.\right.$$
$$\left.\left. + \, \theta \int \mathrm{d}^3x \, \varepsilon^{0ij} \dot{A}_i A_j\right]\right\}. \tag{2.21}$$

In (2.21), if we exponentiate the δ-function representing the Gauss law constraint, we obtain the Chern–Simons term in the action. Further we can obtain the covariant expression by using the Faddeev–Popov trick, passing to a covariant gauge fixing condition.

Table 2. The Dirac brackets among the first-class constraints and the gauge fixing conditions.

	ϕ^0	$\phi^0_{(2)}$	f_1	f_2
ϕ^0	0	0	0	$-2\theta\partial^y_i\partial^i_y\delta(\boldsymbol{x}-\boldsymbol{y})$
$\phi^0_{(2)}$	0	0	$-\partial^x_i\partial^i_x\delta(\boldsymbol{x}-\boldsymbol{y})$	0
f_1	0	$\partial^y_i\partial^i_y\delta(\boldsymbol{x}-\boldsymbol{y})$	0	$-\dfrac{e^2\partial^x_i\delta(\boldsymbol{x}-\boldsymbol{q})\partial^i_y\delta(\boldsymbol{y}-\boldsymbol{q})}{2\theta m}$
f_2	$2\theta\partial^x_i\partial^i_x\delta(\boldsymbol{x}-\boldsymbol{y})$	0	$\dfrac{e^2\partial^x_i\delta(\boldsymbol{x}-\boldsymbol{q})\partial^i_y\delta(\boldsymbol{y}-\boldsymbol{q})}{2\theta m}$	0

3. Exotic spin state

We show that the non-relativistic particle coupled to the Chern–Simons term can have an exotic spin state. We solve the Gauss law constraint. The vector potential is represented by using a multivalued function and the charge density operator of the particle. This is eliminated by rotating a phase of the wavefunction. Then the transformed wavefunction shows exotic behaviour under the operation of the angular momentum operator. So an exotic spin state appears in the system.

3.1. Gauss law constraint

The Gauss law constraint is given by

$$2\theta\varepsilon^{ij}\partial_i A_j = J^0 \tag{3.1}$$

where the charge density operator of the particle is defined as

$$J^0 = -e\delta(\boldsymbol{x} - \boldsymbol{q}). \tag{3.2}$$

In addition, we have the gauge fixing condition

$$\partial_j A^j = 0. \tag{3.3}$$

We solve equations (3.1) and (3.3).

At first, (3.3) gives us

$$A^j = \varepsilon^{jk}\partial_k \phi \tag{3.4}$$

using a scalar function ϕ. Substituting it into (3.1), we obtain

$$\partial_i \partial^i \phi = -\frac{1}{2\theta} J^0 \tag{3.5}$$

which is the Poisson equation in two dimensions. The equation is solved as

$$\phi = -\frac{1}{2\theta}\int \mathrm{d}\boldsymbol{y}\left(-\frac{1}{4\pi}\ln|\boldsymbol{x} - \boldsymbol{y}|^2 + \text{constant}\right) J^0(\boldsymbol{y}). \tag{3.6}$$

So substituting this result into (3.4), we have

$$A^j(\boldsymbol{x}) = \frac{1}{4\pi\theta}\int \mathrm{d}\boldsymbol{y}\,\frac{\varepsilon^{jk}(x-y)_k}{|\boldsymbol{x} - \boldsymbol{y}|^2} J^0(\boldsymbol{y}). \tag{3.7}$$

Equation (3.7) is further rewritten in the form of the total divergence. We introduce an angle variable between the vector $\boldsymbol{x} - \boldsymbol{y}$ and the first axis in two-dimensional space, i.e.

$$\tan\Omega(\boldsymbol{x} - \boldsymbol{y}) = \frac{x^2 - y^2}{x^1 - y^1}. \tag{3.8}$$

Then the relation

$$\partial^j_x \Omega(\boldsymbol{x} - \boldsymbol{y}) = -\varepsilon^{jk}\frac{(x-y)_k}{|\boldsymbol{x} - \boldsymbol{y}|^2} \tag{3.9}$$

holds and using it, (3.7) becomes

$$A^j(\boldsymbol{x}) = -\frac{1}{4\pi\theta}\partial^j_x \int \mathrm{d}\boldsymbol{y}\,\Omega(\boldsymbol{x} - \boldsymbol{y}) J^0(\boldsymbol{y}). \tag{3.10}$$

Here we should comment on the singularity when we set $\boldsymbol{x}$ to $\boldsymbol{q}$ in $A^j(\boldsymbol{x})$. The charge density operator J^0 contains $\delta(\boldsymbol{x} - \boldsymbol{q})$ so that $A^j(\boldsymbol{q})$ is singular. To avoid this difficulty, we regulate the charge density operator as

$$J^0(\boldsymbol{x}) \to J^0(\boldsymbol{x};\varepsilon) = -e\delta_\varepsilon(\boldsymbol{x} - \boldsymbol{q}) = -e\frac{1}{2\pi\varepsilon}\exp[-(\boldsymbol{x} - \boldsymbol{q})^2/\varepsilon]. \tag{3.11}$$

We recognize that $J^0(\boldsymbol{x};\varepsilon)$ is used when $A^j(\boldsymbol{q})$ appears, without referring to this situation explicitly.

3.2. Exotic spin state

The Schrödinger equation of our system is written as

$$-\frac{1}{2m}\left(-i\frac{\partial}{\partial q^i}-eA_i(\boldsymbol{q})\right)\left(-i\frac{\partial}{\partial q_i}-eA^i(\boldsymbol{q})\right)\psi(\boldsymbol{q})=E\psi(\boldsymbol{q}) \tag{3.12}$$

where ψ is the energy eigenstate. The vector potential $A^i(\boldsymbol{q})$ is represented by using the charge density operator as specified by (3.10) with the regularization (3.11). The magnetic field obtained from $A^i(\boldsymbol{q})$ is

$$B(\boldsymbol{q})=\varepsilon^{ij}\partial_i^q A_j(\boldsymbol{q})=\frac{1}{2\theta}J^0=-\frac{e}{2\theta}\delta(0)$$

using (3.1). This is singular everywhere in space. The magnetic flux is

$$\Phi=\int \mathrm{d}^2q\, B(\boldsymbol{q})=-\frac{e}{2\theta}$$

which is finite. The appearance of the singular vector potential means that the configuration space of the system has a non-trivial structure. We can expect a kind of Aharonov–Bohm effect. We can describe the system introducing a multivalued wavefunction absorbing the singular gauge potential by a phase transformation. The transformed wavefunction may describe an exotic state having unusual statistics, which will be presented below.

Consider eliminating $A^i(\boldsymbol{q})$ by changing the phase of the wavefunction as

$$\psi(\boldsymbol{q})\rightarrow \mathrm{e}^{\mathrm{i}\Theta(\boldsymbol{q})}\hat{\psi}(\boldsymbol{q}) \tag{3.13}$$

where the new wavefunction is denoted by the hat symbol. The elimination is achieved when the angle Θ satisfies

$$\frac{\partial}{\partial q_i}\Theta(\boldsymbol{q})=eA^i(\boldsymbol{q}). \tag{3.14}$$

Comparing (3.14) with (3.10), we find that

$$\Theta(\boldsymbol{q})=-\frac{e}{4\pi\theta}\int \mathrm{d}\boldsymbol{y}\ \Omega(\boldsymbol{q}-\boldsymbol{y})J^0(\boldsymbol{y}). \tag{3.15}$$

The Schrödinger equation (3.12) then becomes the term of the free equation which is satisfied with the transformed wavefunction $\hat{\psi}(\boldsymbol{q})$. We should note that this theory is not free because the wavefunction is the multi-valued one.

Now we show that the wavefunction $\hat{\psi}$ describes the exotic spin state. We consider the 2π-rotation of $\hat{\psi}$ in space. In general, if a wavefunction describes a bosonic state, then it does not induce any phase under the 2π-rotation and returns back to the original one after the 2π-rotation. But if a wavefunction corresponds to a fermionic state, the wavefunction changes its sign after the 2π-rotation. Further if there appears any non-trivial phase under the 2π-rotation, the wavefunction describes the exotic spin state. We show that this is the case for the wavefunction $\hat{\psi}$.

For this purpose, we consider the rotation operator which is applied to the wavefunction in two space dimensions. The rotation operator is defined by

$$U_{\mathrm{R}}(\phi) = \mathrm{e}^{-\mathrm{i}\phi L} \tag{3.16}$$

where ϕ is the rotation angle and L is the angular momentum operator defined by

$$L = \varepsilon^{ij} q_i p_j . \tag{3.17}$$

Under the operation of $U_{\mathrm{R}}(\phi)$, the wavefunction ψ behaves as

$$\psi'(\boldsymbol{q}) = U_{\mathrm{R}}(\phi)\psi(\boldsymbol{q}) \tag{3.18}$$

where $\psi'(\mathbf{R}\boldsymbol{q}) = \psi(\boldsymbol{q})$. $\mathbf{R}$ is the 2×2 matrix

$$\mathbf{R} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \tag{3.19}$$

representing the two-dimensional rotation.

Suppose that the wavefunction $\psi(\boldsymbol{q})$ included in the Schrödinger equation (3.12) is the bosonic field. It can be chosen as the eigenstate of the angular momentum operator L with an integer eigenvalue. Thus

$$\psi_l'(\boldsymbol{q}) = U_{\mathrm{R}}(\phi)\psi_l(\boldsymbol{q}) = \mathrm{e}^{-\mathrm{i}\phi l}\psi_l(\boldsymbol{q}) \tag{3.20}$$

with an integer l specifying the quantum number of the angular momentum. For the 2π-rotation

$$\psi_l'(\boldsymbol{q}) = \mathrm{e}^{-\mathrm{i}2\pi l}\psi_l(\boldsymbol{q}) = \psi_l(\boldsymbol{q})$$

so that there is no phase factor as expected.

Let consider the case of the wavefunction $\hat{\psi}(\boldsymbol{q})$. In the same way as (3.20), the rotation of the wavefunction $\hat{\psi}(\boldsymbol{q})$ is given by

$$\hat{\psi}'(\boldsymbol{q}) = U_{\mathrm{R}}(\phi)\hat{\psi}(\boldsymbol{q}) \tag{3.21}$$

where $\hat{\psi}$ is defined in (3.13). We evaluate this as

$$\hat{\psi}'(\boldsymbol{q}) = U_{\mathrm{R}}(\phi)\mathrm{e}^{-\mathrm{i}\Theta(\boldsymbol{q})}\psi(\boldsymbol{q}) = U_{\mathrm{R}}(\phi)\mathrm{e}^{-\mathrm{i}\Theta(\boldsymbol{q})}U_{\mathrm{R}}(\phi)^{-1}U_{\mathrm{R}}(\phi)\psi(\boldsymbol{q}) = \mathrm{e}^{-\mathrm{i}\Theta(\mathbf{R}\boldsymbol{q})}U_{\mathrm{R}}(\phi)\psi(\boldsymbol{q}) \tag{3.22}$$

where we have used the relation

$$U_{\mathrm{R}}(\phi)f(\boldsymbol{q})U_{\mathrm{R}}(\phi)^{-1} = f(\mathbf{R}\boldsymbol{q}).$$

Now we find the typical feature of the phase $\Theta(\boldsymbol{q})$ in evaluating $\Theta(\mathbf{R}\boldsymbol{q})$. Using (3.15) with (3.11), we have

$$\begin{aligned}\Theta(\mathbf{R}\boldsymbol{q}) &= -\frac{e}{4\pi\theta}\int \mathrm{d}\boldsymbol{y}\,\Omega(\mathbf{R}\boldsymbol{q}-\boldsymbol{y})\left(-\frac{e}{2\pi\varepsilon}\right)\exp[-(\boldsymbol{y}-\mathbf{R}\boldsymbol{q})^2/\varepsilon] \\ &= -\frac{e}{4\pi\theta}\int \mathrm{d}\boldsymbol{y}'\,\Omega(\mathbf{R}\boldsymbol{q}-\mathbf{R}\boldsymbol{y}')\left(-\frac{e}{2\pi\varepsilon}\right)\exp[-(\boldsymbol{y}'-\boldsymbol{q})^2/\varepsilon]\end{aligned} \tag{3.23}$$

where we change the integration variable as $\boldsymbol{y} = \mathbf{R}\boldsymbol{y}'$ and use the relation, $\mathbf{R}^{\mathrm{T}}\mathbf{R} = 1$ and $\det \mathbf{R} = 1$. We notice that $\Omega(\mathbf{R}\boldsymbol{q} - \mathbf{R}\boldsymbol{y}') = \Omega(\boldsymbol{q} - \boldsymbol{y}') + \phi$. We then obtain

$$\begin{aligned}\Theta(\mathbf{R}\boldsymbol{q}) &= -\frac{e}{4\pi\theta}\int \mathrm{d}\boldsymbol{y}'\,(\Omega(\boldsymbol{q}-\boldsymbol{y}')+\phi)\left(-\frac{e}{2\pi\varepsilon}\right)\exp[-(\boldsymbol{y}'-\boldsymbol{q})^2/\varepsilon] \\ &= \Theta(\boldsymbol{q}) + \frac{e^2}{4\pi\theta}\phi. \end{aligned} \tag{3.24}$$

Substituting (3.24) into (3.22), we have

$$\hat{\psi}'(\boldsymbol{q}) = \exp\{-\mathrm{i}[\Theta(\boldsymbol{q}) + (e^2/4\pi\theta)\phi]\}U_{\mathrm{R}}(\phi)\psi(\boldsymbol{q}). \tag{3.25}$$

Especially, we choose the eigenstate of the angular momentum operator L with the eigenvalue l as the wavefunction. Then

$$\begin{aligned}\hat{\psi}_l'(\boldsymbol{q}) &= U_{\mathrm{R}}(\phi)\hat{\psi}_l(\boldsymbol{q}) \\ &= \exp\{-\mathrm{i}[\Theta(\boldsymbol{q}) + (e^2/4\pi\theta)\phi]\}U_{\mathrm{R}}(\phi)\psi_l(\boldsymbol{q}) \\ &= \exp\{-\mathrm{i}\phi[(e^2/4\pi\theta) + l]\}\hat{\psi}_l(\boldsymbol{q}). \end{aligned} \tag{3.26}$$

Here we have arrived at the important expression. The novel phase factor $\exp[-\mathrm{i}\phi(e^2/4\pi\theta)]$ appears in addition to the usual phase factor $\mathrm{e}^{-\mathrm{i}\phi l}$. For the 2π-rotation, we have

$$\hat{\psi}_l'(\boldsymbol{q}) = \exp[-\mathrm{i}(e^2/2\theta)]\hat{\psi}_l(\boldsymbol{q}). \tag{3.27}$$

This is just the evidence of the exotic spin. Typically we can realize the following cases:

1. If $e^2/2\theta = 2n\pi$, $\hat{\psi}_l$ is bosonic.
2. If $e^2/2\theta = (2n+1)\pi$, $\hat{\psi}_l$ is fermionic.

Here n is an integer. The other parameter corresponds to the anyon [7]. We should note that case (2) realizes the Bose–Fermi transmutation in quantum mechanics.

4. Relativistic case

We can extend the previous analyses to the relativistic case. Thus the relativistic particle coupled to the U(1) gauge field which has the Chern–Simons term as the action is considered. The discussions are parallel to the previous ones so that we present the results briefly.

4.1. Dirac formalism

The starting Lagrangian is

$$L = -m(1 + \dot{z}_i \dot{z}^i)^{1/2} + \int \mathrm{d}\boldsymbol{x} \, \{A_0(x)\delta(\boldsymbol{x} - \boldsymbol{z}) + eA_i(x)\delta(\boldsymbol{x} - \boldsymbol{z})\dot{z}^i + \theta\varepsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho\}. \tag{4.1}$$

In general, the Lagrangian for the relativistic particle has invariance under the general coordinate transformation. In (4.1), we have already fixed the gauge degree of freedom by imposing $z^0 = x^0 = t$. The dot denotes the time derivative.

The Euler–Lagrange equations are obtained as

$$\frac{\mathrm{d}}{\mathrm{d}x^0}\left(-m\frac{\dot{z}_i}{(1 + \dot{z}_j \dot{z}^j)^{1/2}}\right) + eF_{0i}(x^0, \boldsymbol{z}) + eF_{ji}(x^0, \boldsymbol{z})\dot{z}^j = 0 \tag{4.2}$$

and

$$\theta\varepsilon^{\nu\alpha\beta}\partial_\alpha A_\beta + eg^{0\nu}\delta(\boldsymbol{x} - \boldsymbol{z}) + eg^{i\nu}\delta(\boldsymbol{x} - \boldsymbol{z})\dot{z}_i = 0. \tag{4.3}$$

The canonical momenta are

$$p^i \equiv \frac{\delta L}{\delta \dot{z}_i} = -m\frac{\dot{z}^i}{(1 + \dot{z}_j \dot{z}^j)^{1/2}} + eA^i(x^0, \boldsymbol{z}) \tag{4.4}$$

$$\Pi^\mu \equiv \frac{\delta L}{\delta \dot{A}_\mu} = \theta\varepsilon^{\alpha 0\mu}A_\alpha. \tag{4.5}$$

Equation (4.5) gives the primary constraint

$$\phi^\mu \equiv \Pi^\mu - \theta\varepsilon^{\alpha 0\mu}A_\alpha \approx 0. \tag{4.6}$$

The Poisson brackets are defined as

$$\{A^\mu(\boldsymbol{x}), \Pi^\nu(\boldsymbol{y})\} = g^{\mu\nu}\delta(\boldsymbol{x} - \boldsymbol{y}) \tag{4.7a}$$

$$\{z^i, p^j\} = g^{ij}. \tag{4.7b}$$

By the Legendre transformation, we obtain the canonical Hamiltonian

$$H_{\mathrm{C}} = [m^2 - (p_i - eA_i(\boldsymbol{z}))(p^i - eA^i(\boldsymbol{z}))]^{1/2} - eA_0(\boldsymbol{z}) - 2\theta\int \mathrm{d}\boldsymbol{y}\, \varepsilon^{ij}A_0\partial_i A_j. \tag{4.8}$$

Now, starting from the primary Hamiltonian

$$H_{\mathrm{P}} = H_{\mathrm{C}} + \int \mathrm{d}\boldsymbol{y}\, u_\rho \phi^\rho$$

with the Lagrange multiplier u_ρ, we construct the consistent Hamiltonian system. The requirement of the consistency condition induces the additional constraint

$$\phi^0_{(2)} = e\delta(\boldsymbol{x} - \boldsymbol{z}) + 2\theta\varepsilon^{ij}\partial_i A_j \approx 0 \tag{4.9}$$

which is the Gauss law constraint. Equations (4.6) and (4.9) are all of the constraints.

After this point, the analysis is the same as for the non-relativistic case. We evaluate the Poisson brackets among the constraints. The result is the same as table 1. We classify the constraints into first- and second-class ones. The second-class constraints are not independent of each other. The Gauss law constraint is modified by the linear combination with all of the other second-class constraints so that it becomes the first-class constraint. We compose the Dirac brackets for the dynamical variables. We then arrive at the system described by the total Hamiltonian

$$H_{\mathrm{T}} = [m^2 - (p_i - eA_i(\boldsymbol{z}))(p^i - eA^i(\boldsymbol{z}))]^{1/2} + \int \mathrm{d}\boldsymbol{y}\,(u\phi^0 + v\phi^0_{(2)}) \quad (4.10)$$

and the Dirac brackets

$$\{A_\mu(\boldsymbol{x}), A_\nu(\boldsymbol{y})\}_{\mathrm{D}} = \frac{1}{2\theta}\varepsilon_{0\mu\nu}\delta(\boldsymbol{x}-\boldsymbol{y}) \quad (4.11a)$$

$$\{A_0(\boldsymbol{x}), \Pi_0(\boldsymbol{y})\}_{\mathrm{D}} = \delta(\boldsymbol{x}-\boldsymbol{y}) \quad (4.11b)$$

$$\{A_i(\boldsymbol{x}), \Pi_j(\boldsymbol{y})\}_{\mathrm{D}} = \frac{1}{2}g_{ij}\delta(\boldsymbol{x}-\boldsymbol{y}) \quad (4.11c)$$

$$\{\Pi_\mu(x), \Pi_\nu(\boldsymbol{y})\}_{\mathrm{D}} = \frac{\theta}{2}\varepsilon_{0\mu\nu}\delta(\boldsymbol{x}-\boldsymbol{y}) \quad (4.11d)$$

$$\{z_i, p_j\}_{\mathrm{D}} = g_{ij}. \quad (4.12)$$

4.2. Canonical quantization

As in subsection 2.2, we obtain the equal-time commutators from the Dirac brackets (4.11) and (4.12). The same admissible subsidiary conditions (2.18) and (2.19) are chosen. The situation is the same as the non-relativistic case given in subsection 2.2, except for the total Hamiltonian (4.10).

4.3. Path-integral quantization

The path-integral expression also is obtained in the same manner as in subsection 2.3. The constraints and the gauge fixing conditions are the same as the non-relativistic case. Thus, we have the partition function

$$\begin{aligned} Z = \int \mathrm{d}z_k\,\mathrm{d}p_k\,\mathrm{d}A_i\,\delta(2\theta\varepsilon^{ij}\partial_i A_j(\boldsymbol{x}) + e\delta(\boldsymbol{x}-\boldsymbol{z}))\delta(\partial_i A^i(\boldsymbol{x})) \\ \times \exp\left[i\left(\int \mathrm{d}t\,\{\dot{z}_i p^i - [m^2 - (p_i - eA_i(\boldsymbol{z}))(p^i - eA^i(\boldsymbol{z}))]^{1/2}\}\right.\right. \\ \left.\left. + \theta\int \mathrm{d}^3x\,\varepsilon^{0ij}\dot{A}_i A_j\right)\right]. \end{aligned} \quad (4.13)$$

Of course, this relativistic expression reduces to (2.12) in the non-relativistic limit.

4.4. Exotic spin state

The presence of the exotic spin state is also found in the same way as in section 3. Solving the Gauss law constraint, the gauge field is expressed by using the multivalued function and the charge density operator of the particle. The root of the quantum operator should be understood as it is given by the Taylor series expansion. The gauge field can be eliminated by changing the phase of the wavefunction. Then the transformed wavefunction describes the exotic spin state.

5. Conclusion and discussion

We have considered the quantum mechanics coupled to the U(1) gauge field which has the Chern–Simons term as the action. Following Dirac's prescription, the generalized Hamiltonian formalism has been constructed. The system has been quantized by using the canonical and path-integral methods. We have solved the Gauss law constraint and represented the gauge field by the multivalued function and the charge density operator of the particle. This gauge field has been eliminated by transforming the phase of the wavefunction. We then have shown that the transformed wavefunction describes the exotic spin state. Thus under the 2π-rotation of the wavefunction in space, there appears the novel phase factor. Especially if we choose the physical parameters as satisfying the condition $e^2/2\theta = (2n+1)\pi$, the originally bosonic wavefunction becomes fermionic. This is the quantum mechanical version of the Bose–Fermi transmutation. The other parameter region corresponds to the anyon.

The same analyses have been done in the case of the relativistic particle coupled to the U(1) gauge field with the Chern–Simons term as the action and the exotic spin state has been found in the system.

The essential point is that the gauge field with the Chern–Simons term as the action is represented by using the charge density operator of the particle and the multivalued function which appears in solving the Gauss law constraint. The gauge field does not have the true dynamical degree of freedom and is represented as a singular vector potential. The corresponding magnetic field is also singular though the magnetic flux is finite. These singular behaviours mean that the configuration space of the dynamical variables has non-trivial structure, which is a multi-connected space. Thus we may say that the inclusion of the Chern–Simons term can induce a kind of magnetic field, which may be singular, in the context of the quantization. This situation is naturally expected. The Chern–Simons term is the parity breaking term and may give the magnetic field. The singular gauge field can be absorbed in the wavefunction by the phase rotation. The rotated wavefunction becomes a multivalued function and describes the exotic spin state.

The Chern–Simons term in our model may be considered as a correspondence of a kind of long-range order in a statistical model, which might be a more fundamental model of our model. The transformed wavefunction may be the effective wavefunction adopting such a long-range order. If we can realize the situation where the long-range order becomes dominant, we can see the exotic behaviour which is well described by the transformed wavefunction. Probably, it would be realized in the real physical world as a macroscopic quantum effect.

Based on the analysis presented in this paper, we can proceed with further investigations by application to the real physical system. More extended analysis has been carried out, which will appear elsewhere [24].

Acknowledgments

The author would like to thank Professor T Maskawa for helpful comments and Professor T Inami for useful information on related subjects. He also thanks Dr H Fukutaka, Dr C Itoi, Mr Y Mukaida, and Dr H Yabu for some discussions and informing him some related articles. This work is partially supported by a Grant-in-aid for Scientific Research from the Ministry of Education, Science and Culture (No 63790162) and the fellowship of the Japan Society for the Promotion of Science for Japanese Junior Scientists.

Note added in proof. After this work was completed, we were informed of the preprint [25], in which the multiparticle system coupled to the Chern–Simons term was analysed by using the RPA method. The extension of our study to the multiparticle system is straightforward. In our work, the basis of the quantization of the system is given and the kinematical nature is clarified and, further, the relativistic case is discussed.

References

[1] Jackiw R 1987 *Quantum Field Theory and Quantum Statistics* vol 2 ed I A Batalin *et al* (Bristol: Adam Hilger)

[2] Siegel W 1979 *Nucl. Phys.* B **156** 135
Shonfeld J 1981 *Nucl. Phys.* B **185** 157
S Deser, Jackiw R and S Templeton 1982 *Phys. Rev. Lett.* **48** 975; 1982 *Ann. Phys., NY* **140** 372
C R Hagen 1984 *Ann. Phys., NY* **157** 342; 1985 *Phys. Rev.* D **31** 2135

[3] Dzyaloshinskii I, Polyakov A M and Wiegmann P B 1988 *Phys. Lett.* **127A** 112
Wiegmann P B 1988 *Phys. Rev. Lett.* **60** 821

[4] Polyakov A M 1988 *Mod. Phys. Lett.* A **3** 325

[5] Bednorz G and Müller K A 1986 *Z. Phys.* B **64** 188
Anderson P W 1987 *Science* **235** 1196

[6] Witten E *Preprint* IAS IASSNS-HEP-88/3

[7] Wilczek F 1982 *Phys. Rev. Lett.* **48** 1144; **49** 957
Goldhaber A S 1982 *Phys. Rev. Lett.* **49** 905
Jackiw R and Redlich A N 1983 *Phys. Rev. Lett.* **50** 555

[8] Wu Y-S 1984 *Phys. Rev. Lett.* **52** 2103; **53** 111
Arovas D P, Schrieffer J R, Wilczek F and Zee A 1985 *Nucl. Phys.* **251** 117

[9] Halperin B I 1984 *Phys. Rev. Lett.* **52** 1583 (erratum 2390)
Arovas D P, Schrieffer J R and Wilczek F 1984 *Phys. Rev. Lett.* **53** 722

[10] Wilczek F and Zee A 1983 *Phys. Rev. Lett.* **51** 2250

[11] Belavin A A and Polyakov A M 1975 *JETP Lett.* **22** 245

[12] Bowick M J, Karabali D and Wijewardhana L C R 1986 *Nucl. Phys.* B **271** 417
Semenoff G W 1988 *Phys. Rev. Lett.* **61** 517
Panigrahi P K, Roy S and Scherer W 1988 *Phys. Rev.* D **38** 3199; 1988 *Phys. Rev. Lett.* **61** 2827

[13] Babinovici E, Schwimmer A and Yankielowicz S 1984 *Nucl. Phys.* B **248** 523

[14] Tze C-H 1988 *Int. J. Mod. Phys.* A **3** 1959
Frölich J and Marchetti P-A 1988 *Lett. Math. Phys.* **16** 347

[15] Wen X-G and Zee A 1988 *Phys. Rev. Lett.* **61** 1025
Haldane F D M 1988 *Phys. Rev. Lett.* **61** 1029
Fradkin E and Stone M 1988 *Phys. Rev.* B **38** 7215
Wen X-G, Wilczek F and Zee A 1989 *Phys. Rev.* B **39** 11413

[16] Wen X-G and Zee A 1989 *Phys. Rev. Lett.* **62** 1937, 2873; **63** 461

[17] Deser S and Redlich A N 1988 *Phys. Rev. Lett.* **61** 1541
Aitchison I J R and Mavromatos N E 1989 *Mod. Phys. Lett.* A **4** 521; *Mod. Phys. Lett.* B **3** 1275; 1989 *Phys. Rev. Lett.* **63** 2684
Dunne G V, Jackiw R and Trugenberger C A 1989 *Ann. Phys., NY* **194** 197; 1990 *Phys. Rev.* D **41** 661
Jackiw R 1990 *Ann. Phys., NY* **201** 83

[18] Matsuyama T *Preprint* Research Institute for Fundamental Physics, Kyoto University RIFP-803

[19] Matsuyama T 1989 *Phys. Lett.* **228B** 99

[20] M Lüscher 1989 *Nucl. Phys.* B **326** 557

[21] Matsuyama T 1990 *Phys. Lett.* **144A** 59
[22] Dirac P A M 1964 *Lectures on Quantum Mechanics (Belfer Graduate School Monograph Series 2)* (New York: Yeshiva University)
Sundermeyer K 1982 *Constrained Dynamics (Lecture Notes in Physics 169)* (Berlin: Springer)
[23] Faddeev L D 1970 *Theor. Math. Phys.* **1** 1
Senjanovic P 1976 *Ann. Phys., NY* **100** 227
[24] Matsuyama T 1990 in preparation
[25] Chen Y-H, Wilczek F, Witten E and Halperin B I 1989 *Int. J. Mod. Phys.* B **3** 1001